GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

48. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

The Simson line belonging to one point of intersection of Brocard's Diameter of a triangle with the circumcircle of this triangle, is either parallel or perpendicular to the besector of the angle formed by the side BC of the triangle ABC and the corresponding side $B'C'$ of Brocard's triangle.

Solution by the PROPOSER.

We shall first prove the following lemma:

1. If upon the sides of the triangle ABC are constructed similar isosceles triangles BA_2C, CB_2A, and AC_2B, and if the perpendicular A_2M_a is produced below BC, so that A'_2M_a is equal to A_2M_a then is $AC_2A'_2B_2$ a parallelogram.

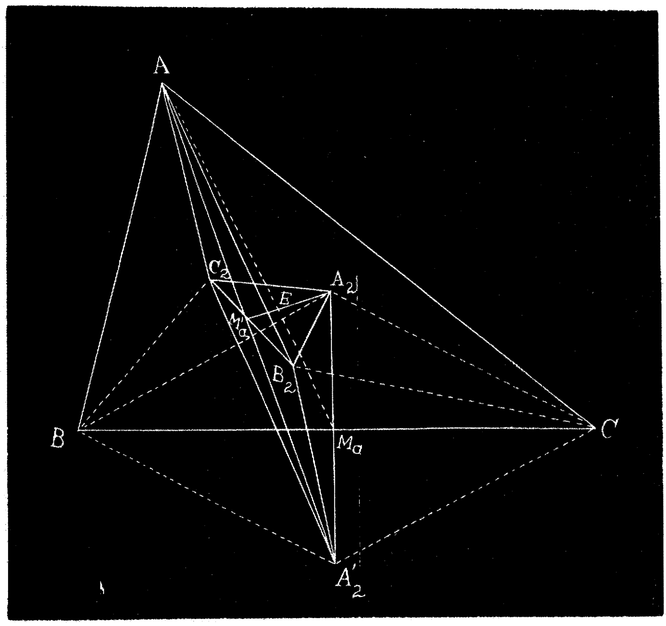

Fig. 1.

$$\measuredangle C_2BA'_2=\measuredangle C_2BC+\measuredangle CBA'_2\ ;$$

but $\quad \measuredangle CBA'_2=\measuredangle A_2BC\ ;$

therefore $\quad \measuredangle C_2BA'_2=\measuredangle C_2BC+\measuredangle A_2BC\ ;$

but $\quad \measuredangle A_2BC=\measuredangle C_2BA,$

hence $\quad \measuredangle C_1BA'_2=\measuredangle C_2BC+\measuredangle C_2BA=\measuredangle ABC.$

The triangle A_2BM_a is similar to triangle C_2BM_c (since they are right triangles having $\measuredangle A_2BC = \measuredangle C_2BA$).

$$\text{Therefore } A_2B : C_2B = BM_a : BM_c = \frac{a}{2} : \frac{c}{2} = a : c\,;$$

but $\quad A_2B = A'_2B\,;$
hence $\quad A'_2B : C_2B = a : c,$
and since the $\measuredangle A'_2BC_2 = \measuredangle ABC$, therefore is triangle A'_2BC_2 similar to triangle ABC. In a similar manner can be proved that the triangle $B_2CA'_2$ is also similar to triangle ABC, and therefore A'_2BC_2 and $B_2CA'_2$ are similar to one another. But $A'_2B = A'_2C$ and consequently are the triangles A'_2BC_2 and $B_2CA'_2$ not only similar but also equal and therefore $B_2A'_2 = C_2A$. In a similar manner can be proved that $AB_2 = C_2A'_2$ or $AC_2A'_2B_2$ is a parallelogram.

2. The triangles ABC and $A_2B_2C_2$ have the same median point E.

Since $AC_2A'_2B_2$ is a parallelogram, the diagonals AA'_2 and A_2C_2 will bisect each other at the point M'_a. $A_2M'_a$ is a median line in the triangle $A_2B_2C_2$ as well as in the triangle $AA_2A'_2$. A second median line in the triangle $AA_2A'_2$ is AM_a (since $A_2M_a = A'_2M_a$); we have, therefore, that $A_2E = 2EM'_a$ and $AE = 2EM_a$. But AM_a is also a median line in the triangle ABC, therefore is E the median point in the triangle ABC as well as in the triangle $A_2B_2C_2$.

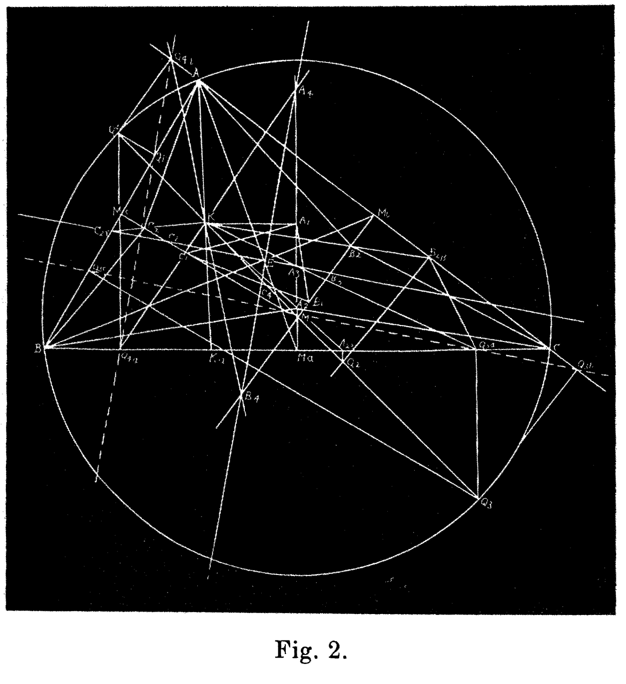

Fig. 2.

A_2, B_2, and C_2 were the vertices of similar isosceles triangles constructed upon the sides of the triangle ABC, and let KA_2, KB_2, and KC_2 meet the sides BC, AC, and AB respectively at $A_{2\alpha}$, $B_{2\beta}$, and $C_{2\gamma}$, then it can be proved that triangle $A_{2\alpha}\ B_{2\beta}\ C_{2\gamma}$ is similar to triangle $A_2\ B_2\ C_2$, the center of similitude being K. If we erect a perpendicular at $A_{2\alpha}$ to BC to meet Brocard's Diameter at Q_2, then, putting for A_1M_a, B_1M_b, their equals KK_a, KK_b respectively, ($A_1B_1C_1$ is Brocard's triangle), we have

$$\frac{A_2M_a}{KK_a}=\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{Q_2M}{Q_2K}.$$

Since the triangles A_2BC and B_2AC are similar, we have

$$\frac{A_2M_a}{B_2M_b}=\frac{M_a\,C}{M_b\,C}=\frac{a}{b}=\frac{A_1M_a}{B_1M_b},$$

or
$$\frac{A_2M_a}{A_1M_a}=\frac{B_2M_b}{B_1M_b}=\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K}.$$

Therefore
$$\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{B_{2\beta}B_2}{B_{2\beta}K}.$$

Similarly we get
$$\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{C_{2\gamma}C_2}{C_{2\gamma}K}=\frac{Q_2M}{Q_2K},$$

or, triangles $A_2B_2C_2$ and $A_{2\alpha}B_{2\beta}C_{2\gamma}$ are similar, and K is the center of similitude. From the equation

$$\frac{B_{2\beta}Q_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K},$$

it follows that $B_{2\beta}Q_2$ is parallel to B_2M, and since B_2M is perpendicular to AC, therefore $B_{2\beta}Q_2$ is also perpendicular to AC, or the perpendicular at $B_{2\beta}$ to AC passes through Q_2. Similarly, the perpendicular at $C_{2\gamma}$ to AB passes through Q_2. If, now, Q_2 is made to coincide with either Q_3 or Q_4, the points of intersection of Brocard's Diameter and the circumcircle of the triangle ABC, the triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ will then degenerate into the straight lines $Q_{3a}\,Q_{3b}\,Q_{3c}$ and $Q_{4a}\,Q_{4b}\,Q_{4c}$ which are the Simson lines belonging to Q_3 and Q_4 with respect to the circumcircle of the triangle. The triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$ and $A_4B_4C_4$, which will be parallel to the Simson lines belonging to Q_3 and Q_4; and they will pass through the median point E, for the lines $A_3B_3C_3$ and $A_4B_4C_4$ still have the median point E in common with ABC.

Also, A_3, B_3, C_3 and A_4, B_4, C_4 are on the perpendiculars at the middle points of the respective sides of the triangle ABC. Since the Simson lines to Q_3 and Q_4 correspond to the extremities of a diameter, they are perpendicular to each other, and therefore their parallels $A_3B_3C_3$ and $A_4B_4C_4$ are also perpendicular to each other.

Furthermore, $Q_{3a}M_a=Q_{4a}M_a$,

$$Q_{3a}M_a: M_aK_a=Q_{4a}M_a: M_aK_a,$$

$$Q_{3a}M_a: M_aK_a=Q_{3a}A_3 : A_3K=A_3M_a: A_3A_1,$$

and

$$Q_{4a}M_a: M_aK_a=Q_{4a}A_4 : A_4K=A_4M_a: A_4A_1,$$

or

$$A_3M_a: A_3A_1=A_4M_a: A_4A_1,$$

whence $\{M_aA_1,\ A_3A_4\}$ is an harmonic range, and $E\{M_aA_1,\ A_3A_4\}$ is an harmonic pencil. Since $\measuredangle A_4EA_3=90^\circ$, EA_3 will bisect the angle A_1EM_a.

Now, in two similar triangles the bisectors of the angles formed by any line in one triangle with the corresponding line in the other triangle are parallel to each other, hence the bisector of the angle formed by A_1E and EM_a, or the line AE, *i. e.*, the line $A_3B_3C_3$,—which is parallel to the Simson line belonging to one of the points of intersection of Brocard's Diameter, and the circumcircle about the triangle ABC,—is parallel to the bisector of the angle formed by B_1, C_1, and BC. (For particulars I can refer to my Geometrical Treatment of curves which are isogonal conjugate to a straight line with respect to a triangle, published by Leach, Shewell and Sanborn, New York.)

An excellent solution of this problem was also received from Professor G. B. M. Zerr.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsalvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

III. Solution by the PROPOSER.

The $\triangle$ BDE has each side$=\sqrt{2}a$, hence the radius of its circumscribed circle$=\frac{1}{3}\sqrt{6}a$. Hence the distance of A to the plane of $BDE=\frac{1}{3}\sqrt{3}a$. Take the origin at the center of the cube and the line AG as the axis of Z. The revolution will bring each line of the gauche hexagon $EHDCBF$ into either